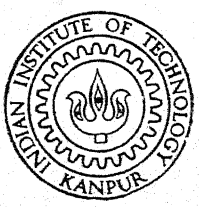
RESPONSE ANALYSIS OF STRUCTURES UNDER RANDOM VIBRATION ENVIRONMENT—A MONTE CARLO APPROACH

By
P. L. N. MURTHY



DEPARTMENT OF AERONAUTICAL ENGINEERING
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RESPONSE ANALYSIS OF STRUCTURES UNDER RANDOM VIBRATION ENVIRONMENT—A MONTE CARLO APPROACH

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P. L. N. MURTHY

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CERTIFICATE

Certified that this work 'Response Analysis of Structures Under Random Vibration Environment - A Monte-Carlo Approach' has been carried out under my supervision and that the work has not been submitted elsewhere for a degree.

(N.C. Nigam)
Associate Professor
Department of Aeronautical Engineering
Indian Institute of Technology, Kanpur

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ABSTRACT

A Monte-Carlo simulation technique suggested by Shinozuka is used to investigate the response characteristics of a few simple linear and non-linear structures, under random vibration environment. Stationary response characteristics are computed for both linear and nonlinear systems. For the linear system the transient response is also computed.

Results of simulation are used to calculate the first-passage probability density, the expected amount of time spent above a prescribed threshold, the number of crossings of a level per unit time. A linear single degree of freedom system is considered for solving the above mentioned problems.

The random walk model developed by Toland is extended to a two degree of freedom system, and the computation problems involved in solving such a model are discussed.

NOMENCLATURE

A	number of reverse arrangements
a, b	barriers defining the safe domain
B_e	resolution Band width
C	roughness constant
c, c_1, c_2	damping coefficients
E	expectation operator
e	exponential
$F(t)$, $F_1(t)$ $F_2(t)$	random force
f_c	cut off frequency or nyquist frequency
$f_1(t)$, $f_2(t)$	generalised forces
$G_o, G(\omega)$	one sided power spectral densities
$g(t,\tau)$	probability density function of first passage time
$H(w)$	frequency response function
$h(t)$	impulse response function
k, k_1, k_2	stiffnesses
L	safe run
M_1, m_1, m_2	masses
N	sample size
$p_Y(y)$	probability density function

$R_X^{\cdot\cdot}(t_1, t_2)$, $R_Y^{\cdot\cdot}(t_1, t_2)$	auto correlation functions
$x(t)$, $y(t)$	random processes
$\bar{x}(t)$	sample function of random process
w_0	natural frequency (rad/sec.)
w_k	random frequency
ζ, ζ_1, ζ_2	fraction of critical damping
Ω	spatial frequency (rad/ft.)
σ_x, σ_y	r.m.s. values of x and y
Δ^*	grid size in the velocity direction
Γ	safe domain
ϕ_k	random phase angle
Ω_1, Ω_2, Ω_4...	transition probability functions of order 1, 2 and 4 respectively.

CONTENTS

CHAPTER I

INTRODUCTION

I.1 In the past two decades there has been significant development in the theory of Stochastic Processes. This has made it possible to predict the dynamic structural response under random loading conditions. It has been studied extensively with highly idealised load structure systems. However, the real life loading conditions and structural systems differ significantly from the assumed deterministic models. Loading conditions and material properties in almost all real life structures have a large measure of uncertainity requiring a stochastic approach for arealistic treatment. Hence a probabilistic approach provides a more realistic approach to assess the overall structural performance.

The probabilistic approach too has its limitations. Loading conditions, many a time, are non-stationary in nature. The well developed principles of stationary stochastic processes cannot be applied to nonstationary stochastic processes. Also it is limited to the linear structural models. Attempts have been made to solve some of the problems, which resulted in the equivalent linearization techniques and perturbation techniques. However, these are restricted to structures with weak non-linearity. Some of the problems such as the first passage probability and fatigue damage of a structure, which form the

more important failure criteria, are yet unsolved. Monte Carlo techniques have emerged out as powerful tools and have been used successfully to solve the above mentioned problems. These are essentially numerical techniques. Some of the problems which are not amenable to conventional mathematical analysis are: a) analysis of panel vibrations of aircraft and submarines induced by boundary layer turbulence, b) analysis of off shore structures taking the wave, structure interaction into consideration, c) analysis of vehicle and aircraft vibration caused by the roughness of the ground etc.

The Monte Carlo techniques basically consists of simulating the loading conditions, which have fixed data and solving the resulting equation of motion using numerical integration. The artificial data have the same statistical properties as that of parent one. For a stationary stochastic process a single sample response function is enough to compute the statistics such as mean, variance etc., as the principle of ergodicity is valid. The temporal averages can substitute ensemble averages. Wherever the stochastic process of interest is nonstationary, an ensemble of sample functions can be collected and the averages can be computed across the ensemble. Thus by using artificial data for the loading conditions statistics such as peak response distribution, number of times a critical value is attained, transient behaviour of the structures, can be conveniently computed following the Monte Carlo approach. The simulation

techniques have the advantage over the conventional methods in that these can be applied to any type of load structure systems.

The limitation of Monte Carlo approaches is that often they require a considerable amount of computer time. Analysis of non-stationary vibrations require large sample size. Particularly for the structures with low damping, it requires a very large number of sample functions. For the existing computer system, IBM 7044, the time consumed is of the order of a few hours. However, the advent of third generation computers should overcome this problem.

I.2. Literature Survey

I.2.1 The basic principles of stochastic process theory and its application to the response analysis of structural systems have been discussed extensively by Lin [1] and Crandall [2] in their works. Both time and frequency domain approaches are elaborated to deal with the response characteristics. But these methods become extremely combursome and involved when dealing with the response characteristics of complicated systems especially subjected to non-stationary excitations. Monte-Carlo simulation techniques have been proved to be extremely useful in dealing with such cases [6,7,8,9].

Filtering techniques were being used by many authors before Borgman [5] published an efficient means of simulating

a sample function of a stochastic process. He considered the ocean surface elevation as a random process and represented it as a sum of a series of cosine functions with random phase angles and amplitudes consistent in the energy in the target spectral density at that frequency.

Later Shinozuka [3,4] proceeding in the similar lines has developed a more generalised approach to simulation, which can be applied to multi-variate and multi dimensional processes. Also it has been shown that among all possible series expansions of a random process, Shinozuka's approach gives the best approximation.

A number of such diverse problems like, the nonlinear panel response from a turbulent boundary layer [7], nonlinear dynamic analysis of off shore structures [8], response of the structures to wind loads [9], have been solved recently following the Monte-Carlo simulation technique of Shinozuka.

I.2.2 Transient Response Analysis

The transient response of a system under random excitation has a history extending over the past 40 years. Uhlenbeck and Ornstein gave the first solution to the problem in 1930. Later Uhlenbeck and Wang [15] have solved the problem using the Fokker-Planck equation for the system. Subsequently, several authors have considered the response of dynamic

systems confining themselves to the stationary aspects of the motion. Caughey and Stumpf [11] have analysed first, the transient motion of a single degree of freedom system, subjected to a stationary random input process, having an arbitrary power spectrum. In the same reference an application to the response analysis of a structure subjected to strong motion earthquake is given. Since that time, a significant amount of work has been contributed by several authors, mainly in the area of earthquake engineering, dealing with the non-stationary transient, response analysis of the structures subjected to earthquake accelerations. However, the simulation techniques have dominated, whence a number of sample response functions were simulated artifically and the response characteristics were computed from the ensemble averages.

Recently, Chakrovorty [10] has developed an analytical approach following the evolutionary spectrum concept of Priestly [21], for the non-stationary response analysis of linear structures subjected to the white noise excitation. In the same reference, an example is given where the transient motion of an equipment in a building which subjected to earthquake acceleration is studied. The support acceleration of the equipment becomes non-stationary because of its transient nature.

1.2.3 First Passage Problems

First excursion failure probability and the fatigue damage are two important modes of failure in the area of random vibrations. The former relates to the time at which for the first time the response of a structure crosses a specified threshold and the later to the cumulative damage during vibration. So far the exact solution even for the simplest case i.e. that of a narrow band random process such as the output of a lightly damped linear oscillator, has not been established. However, based on the numerical results certain analytical approaches have been suggested which could establish the bounds for such probabilities. Following is a brief survey of the previous work in this area.

Some of the first passage problems for a lightly damped linear oscillator excited by a white noise have been solved by Crandall, Chandiramani and Cook [13] in 1966. They described two numerical approaches to the problem. In one the probability statistics of safe intervals were made use of to solve the problem and in the other a markov vector approach was used. Two types of initial conditions, the zero start and the stationary start and three different types of safe domains the single barrier, the double barrier and the envelope type of barrier were considered.

Crandall [14] has described variety of analytical approximations to the first passage probability density for

linear oscillator and compared with results obtained by simulation and numerical work. He derived expressions for the first passage probability density with the following assumptions:

1) Assumption of independent crossings
2) Assumptions of independent peaks
3) Assumption of independent envelope crossings
4) Assumption of independent envelope peaks
5) Two state Markove Process assumption.

Ariaratnam and Hennon [17] have solved the first passage problem associated with snap - through of a shallow cylindrical shell subjected to a wide band stationary stochastic loading. They followed the numerical approaches given in Ref. [13].

Recently Toland, Yang and Hsu [16,20] have proposed a random walk model for the linear oscillator and solved the first passage problem. The details of the model are discussed in the Chapter V.

I.3 The Present Thesis Work

In the present thesis work the response characteristics of a single degree of freedom linear system to band limited white noise excitation are computed to illustrate the Monte Carlo simulation techniques. As an application it is extended to the response analysis of an aircraft-landing gear assembly to taxiing induced vibrations due to runway

undulations. Briefly the contents of the thesis are the following:

A simulation procedure to generate sample functions of any stationary gaussian random process, with prescribed statistical properties is presented in Chapter II. Chapter III contains an analytical treatment, which makes the use of evolutionary spectrum concept, for the analysis of non-stationary, transient response of the linear system, followed by a numerical approach. Also a Monte-Carlo approach to compute the stationary response characteristics of a single degree of freedom system, an extension to a nonlinear system of Duffing's type and the stationary response analysis of an aircraft-landing gear assembly during taxiing and take off are given. In the Chapter IV the first passage probability density and other related statistics such as the expected amount of time spent above a prescribed level, the expected number of crossings of a level, mean time to failure are discussed. A random walk model for the analysis of first excursion failure probabilities is presented in Chapter V.

CHAPTER II

SIMULATION

II.1 Most structural systems operate in random environment. To predict the dynamic structural performance, therefore, either a sufficient number of sample functions of the environment should be available or such sample functions may be simulated based on available stochastic properties. This chapter presents a brief discussion of the simulation procedure employed for the present thesis work.

II.2. Two basic approaches are available to simulate a random process: 1). The required random process is obtained as the output of a linear system having an appropriate frequency response function and is modulated by an envelope function. These filtering techniques dominated for quite a long time, though they are limited to a one-variate, one dimensional random process. 2) The second approach is based on superposition of a series of sinusoidal pulse shapes each having a random phase and an amplitude consistent with the target power spectral density. It is Borgman [5] who first simulated the ocean surface elevation following the above approach. This has an advantage over the previous one in that it can be extended to multivariate, multi dimensional random processes also. Later Shinozuka [3], [4], developed the wave superposition method for simulating both stationary and non-stationary Random Processes.

Shinozuka's method has been followed in the present work. The governing equations are as follows:

A homogeneous, Gaussian, one variate and one dimensional random process, $x_0(t)$, with zero mean, and spectral density function, So (w) can be expressed in the form of the sum of the cosine functions by

$$x_o(t) = \sqrt{2} \sum_{k=1}^{N} A_k \cos (w_k t - \phi_k) \qquad \text{(II.1.1)}$$

where ϕ_k are random variables distributed uniformly between 0 and 2π, and are independent.

$$A_k = [G_o (w) \Delta w]^{1/2} \qquad \text{(II.1.2)}$$

$$w_k = (k-1/2)\Delta w \qquad \text{(II.1.3)}$$

$G_o(w)$ is the one sided power spectral density and is related to the mathematical two sided P.S.D. by

$$G_o(w) = 2 S_o(w)$$

A sample function $\bar{x}(t)$ of the random process x(t) can be generated digitally on a computer, by replacing the random variables ϕ_k by their realised values.

$$\bar{x}(t) = \sqrt{2} \sum_{k=1}^{N} A_k \cos (w_k t - \phi_k) \qquad \text{(II.1.4)}$$

The normality of the simulated process is ensured by virtue of central limit theorem [1]. It has been shown in

Ref. [4] that the temporal and ensemble averages coincide, at least upto second moment, in limit to the target values. This establishes the ergodicity. This makes the method directly applicable to a time domain analysis where the ensemble averages can be evaluated in the form of temporal averages.

II.2. Tests for Stationarity:

In this section two non-parametric statistical tests are described by means of which the stationarity of the simulated random process can be tested.

Once a sample function of a stationary random process has been generated, it is of considerable interest to know how far the simulated sample deviates from stationarity and at what level of significance the data can be trusted. Stationarity implies that all statistical properties of the stochastic process of interest are invariant with respect to time transactions. Under following assumptions, practical tests for testing stationarity can be developed. [REF 22]

1) For a stationary stochastic process, if we compute the statistical properties such as means and mean square values, by time averaging over a sequence of short time intervals of a sample record, the values would not vary too significantly from each other.

2) It is enough to test for weak stationarity for verifying strong stationarity.

3) The length of the sample record is very long compared to the random fluctuations of the data time history.

4) If the mean square value is stationary then the auto correlation function of the data is also stationary.

The sample record is divided into a number of intervals and the mean and mean square values are computed to form sequences as follows:

$$\bar{x}_1, \bar{x}_2, \bar{x}_3 \ldots \bar{x}_n$$
$$\bar{x}_1^2, \bar{x}_2^2, \bar{x}_3^2 \ldots \bar{x}_n^2$$

The above sequences are tested for underlying trends or variations other than due to expected sampling variations. The tests are described below.

Run Test [22]:- A run is defined as a sequence of identical observations that are followed or preceded by a different observation or no observation at all. Let the sequence of measured values x_i, i=1, 2...N has a mean $\bar{x}$. Suppose we compare each x_i with $\bar{x}$ and form a sequence of plus and minus observations, plus indicating that $x_i \geq \bar{x}$ and minus, $x_i < \bar{x}$, we might get a sequence as follows:

+ +	- - -	+ 1	——	+	. . . .	+ +
1	2	3	4			N

From the sequence the number of runs is counted as indicated. The number of runs occur in a sequence gives an indication whether the values x_i are independent or biased. The distribution function of r(k) is presented in Table 4.11, page 170, of Ref.[22]. The hypothesis that the simulated sample function is stationary can be tested by observing the number of runs in the sequence of means, and mean square values at a desired level of significance.

Trend Test:-

It is similar in application to run test. Suppose we have a sequence x_i, i = 1, ...N. The number of times that $x_i > x_i$ for i<j is counted. Each such inequality is called a reverse arrangement. The total number of reverse arrangements is denoted by A.

A is defined mathematically as follows. For the set of observations $x_1, x_2 \cdots x_n$, define

$$h_{ij} = \begin{cases} 1 & x_i > x_j \quad i<j \\ 0 & \text{otherwise} \end{cases}$$

then $$A_i = \sum_{j=i+1}^{N} h_{ij} \quad \cdots\cdot a \qquad \text{(II.2.1)}$$

and $$A = \sum_{i=1}^{N-1} A_i \quad \cdots\cdot b$$

After counting the number of reverse arrangements in a sequence, the hypothesis can be tested at any desired

level of significance, by checking whether it is falling within the corresponding limits specified in the standard tables [Table 4.12, page 171, Ref.[22]].

II.3. Power Spectral Density Calculations

In this section, a method of estimating the power spectral density, and choice of various parameters involved are discussed.

Def. Power Spectral Density of a stochastic process is defined as the Fourier Transform of the auto correlation function as follows.

$$G_x(f) = 2 \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j2\pi f\tau}\, d\tau \qquad \text{(II.3.1)}$$

where $R_x(\tau) = E\,[x(t)\, x(t+\tau)]$

$G_x(f)$ is one sided power spectral density and is related to the mathematical two-sided power spectral density as follows.

$$S_x(f) = S_x(-f) = G_x(f)/2 \qquad \text{(II.3.2)}$$

In order to estimate the power spectral density from the simulated sample function of the random process of interest, Eq. II.3.1 should be discretized. It can be easily shown that [22]

$$\bar{G}_k = \bar{G}_x\left(\frac{kfc}{m}\right) = 2h\left[\hat{R}_o + 2\sum_{r=1}^{m-1} \hat{R}_r \cos(\pi rk/m) + (-1)^k \hat{R}_m\right] \qquad \text{(II.3.3)}$$

which gives a raw estimation of the P.S.D. function at discrete frequency intervals $(\frac{kfc}{m})$, k = 1, 2, ...m.

where fc is cut off frequency and is given by $fc = \frac{1}{2h}$

h is sampling interval

R_r is estimate of correlation function at lag r

m is maximum lag number

The parameters h, m and fc should be chosen appropriately to get a good estimate of P.S.D. function. It can be shown that the normalized standard error involved in the estimation of P.S.D. from Eqn. II.3.3 is given by

$$\epsilon^2 = m/N \qquad \text{(II.3.4)}$$

where N is the sample size.

The maximum lag number m, is also related to the resolution band width Be of the P.S.D. estimate by

$$mh = 1/Be \qquad \text{(II.3.5)}$$

Thus a large m, though increases the resolution of the P.S.D. estimate, will increase the uncertainity of the estimate. So a compromise should be arrived in selecting the value of m such that sufficient resolution with minimum statistical uncertainity is ensured. Usually m is chosen less than one tenth of the sample size N.

The auto correlation function $\hat{R}_r$ is estimated by

$$\hat{R}_r = \hat{R}_x(rh) = \frac{1}{N-r} \sum_{m=1}^{n-r} x_n x_{n+r} \qquad \text{(II.3.6)}$$

$$r = 0, 1, 2, \ldots m$$

As the value of h is usually fixed before hand, the value of cut-off. frequency, frequency beyond which the P.S.D. has no significant value, should be set at least equal to 1/2h, which is called the Nyquist frequency.

$$fc \geq 1/2h \qquad \text{(II.3.7)}$$

$\bar{G}_x(f)$ will be only a raw estimate and it should be smoothed to get a better estimate of the true P.S.D. This can be done by multiplying the auto correlation function with a 'Lag Window', which tantamounts to weighing the correlation function non-uniformly. The final formula is

$$\hat{G}_k = \hat{G}_x\left(\frac{kfc}{m}\right) = 2h\left[\hat{R}_o + \sum_{r=1}^{m-1} D_r \hat{R}_r \cos\frac{\pi rk}{m}\right] \qquad \text{(II.3.8)}$$

where $D_r = 1 + \cos\frac{\pi r}{m}$ $\quad r = 0, 1, 2, \ldots m$

$r > m$ $\qquad \text{(II.3.8(a))}$

for Hanning's Lag Window.

and,

$$D_r = 1+6\left(\frac{r}{m}\right)^3 - 6(r/m)^2, \quad r = 0, 1, 2, \ldots m/2$$

$$= \left[1 - \left(\frac{r}{m}\right)\right]^3 \ldots\ , \quad r = \frac{m+1}{2}, \ldots m$$

$$= 0 \qquad r > m \qquad \text{(II.3.8(b))}$$

for Parzen's Lag Window.

Any one of the above two lag windows can be used satisfactorily, but it is found from the results that for power spectral density estimates, Parzen's lag window gives a smoother and better estimate than Hanning's lag window. The effect of varying m is also studied. It is found that as m increases the scatter in P.S.D. estimate increases.

II.4 Results and Discussion:

Presented herein are the results of the simulation technique described in the previous paragraphs. Two examples are considered. For the first one a wide band random process with constant spectral density is chosen. The target power spectral density function is given by the following equation.

$$G_o(w) = 4 \text{ units} \qquad w_1 < w < w_2 \qquad \text{(II.4.1)}$$

where w_1 = lower limit for frequency = 0 cps

w_2 = cut-off frequency = 100 cps

No. of terms considered for the simulation are 100.

Second example is that of runway roughness for which the power spectral density function may be written as

$$G(\Omega) = \frac{C}{\Omega^2 + \gamma^2} \qquad \text{(II.4.2)}$$

$$\Omega_1 < \Omega \leq \Omega_2$$

where Ω = wave number in rad/ft.

γ = .001791 rad/ft.

γ is introduced in the equation II.4.2 to avoid the singularity at the origin and to make the power spectral density finite at the origin.

C = a constant varying from $.5 \times 10^{-6}$ to 20×10^{-6}

$$\Omega_1 = 0$$

$$\Omega_2 = 1/2h.$$

where h is the step length used for the integration of the equation of motion.

Sample functions are generated according to the Equation II.1.4 where

$$w_k = (k-1/2)\ \Delta w + w_1$$

$$K = 1, 2 \ldots N$$

and $\Delta w = (w_2 - w_1)/N$

ϕ_k are generated uniformly between '0' and '2π' making use of the built in subroutine RNDY1 of IBM 7044 system.

In Table No.II.1 the results of tests for stationarity are presented. The number of runs and the number of reverse arrangements clearly fall within the respective acceptable regions thus establishing the stationarity of the generated sample.

Results of power spectral density estimates are presented in Figs. II.1 and II.2. Table No.II.2 gives a comparison of the power spectral density estimates from a sample function, using Hanning's lag window and Parzen's lag window. It may be infered from Fig.II.1 that Parzen's lag window gives a smoother estimate. From the

Fig. II.2 it can be seen that the scatter in the estimate of power spectral density increases with lag number m, which follows from Eq. II.3.2. For m equal to 20 the bounds are $4 \pm .2832$ and for m equal to 80 the corresponding limits are $4 \pm .5664$.

CHAPTER III

RESPONSE ANALYSIS

Any structural system when subjected to an excitation undergoes a period of transient vibration. If the excitation is a stationary random process, the structural response eventually becomes stationary. However, the transient response is non-stationary. This chapter provides a brief discussion of an analytical approach [10], which uses the concept of evolutionary spectrum to find the transient response of a simple linear oscillator subjected to white noise excitations and a numerical technique based on the ensemble averages, a Monte-Carlo technique for finding the stationary response characteristics of a linear and nonlinear single degree of freedom systems and an idealised model of an aircraft - landing gear assembly. Numerical approach utilises the results of simulation.

2. The characteristics of a linear time invariant system, which is shown in Fig.III.1 can be represented by the well known differential equation

$$\ddot{y}(t) + 2\zeta w_o \dot{y}(t) + w_o^2 y(t) = -\ddot{x}(t) \qquad \text{(III.2.1)}$$

where $w_o^2 = k/m$

$\zeta = c/2mw_o$

w_o is the natural frequency, c is the viscous damping coefficient, ζ is the fraction of critical damping, $\ddot{x}(t)$ is the base excitation in terms of acceleration and y(t) is relative displacement.

The auto correlation function of $\ddot{x}(t)$ may be written as

$$R_{\ddot{x}}(t_1,t_2) = E[\ddot{x}(t_1)\ \ddot{x}(t_2)] \qquad \text{(III.2.2)}$$

When the process is weakly stationary the auto correlation function can be written as

$$R_{\ddot{x}}(\tau) = \int_{-\infty}^{\infty} S_{\ddot{x}}(w)\ \exp[i w (t_2-t_1)]\ dw \qquad \text{(III.2.3)}$$

where

$\tau = t_2 - t_1$ and $S_{\ddot{x}}(w)$ - two sided power spectral density of input Random Process.

The response y(t) is non-stationary, as it builds up from rest, and its auto correlation function may be written

$$R_y(t_1,t_2) = E[y(t_1)\ y(t_2)] \qquad \text{(III.2.4)}$$

The response of the oscillator y(t) is given by the convolution of the impulse response function with input excitation $\ddot{x}[t]$. Thus,

$$y(t) = \int_0^t \ddot{x}(t)\ h(t-\tau)\ d\tau \qquad \text{(III.2.5)}$$

where h(t) is the impulse response function and $h(t-\tau) = 0$, for $t < \tau$.

Substituting Eq. III.2.5 in Eq. III.2.4,

$$R_y(t_1,t_2) = E\Big[\int_0^{t_1} \ddot{x}(t_1)\, h(t_1-\tau_1)\, d\tau_1 \int_0^{t_2} \ddot{x}(t_2) h(t_2-\tau_2)\, d\tau_2\Big] \qquad \text{(III.2.6)}$$

After simplifying the Eq. III-2.6, we get the following relation for $R_y(t_1,t_2)$,

$$R_y(t_1,t_2) = \int_{-\infty}^{\infty} S_{\ddot{x}}(w)\, K(w,t_1)\, K^*(w,t_2)\, \exp[iw(t_1-t_2)]\, dw \qquad \text{(III.2.7)}$$

where $K(w,t) = \int_0^t h(u)\, \exp(-iwu)\, du$ (III.2.8)

and $K^*(w,t)$ is the complex conjugate of $K(w,t)$. $K(w,t)$ is analogous to the frequency response function and is called time dependent transfer function.

The impulse response function $h(t)$ and the frequency response function $H(w)$ are given by [1]

$$h(t) = -\frac{1}{w_o\sqrt{1-\zeta^2}} \exp[-\zeta w_o t] \sin(w_d t);\ t \geq 0$$

$$= 0;\quad t < 0 \qquad \text{(III.2.9(a))}$$

$$H(w) = -(w_o^2 - w^2 + 2i\zeta w_o w)^{-1} \qquad \text{(III.29(b))}$$

$K(w,t)$ may be obtained as [REF. 10]

$$K(w,t) = H(w)\left\{1-\left[\cos w_d t + \frac{\zeta w_o + iw}{w_d}\sin w_d t\right]\exp[-(\zeta w_o + iw)t]\right\} \qquad \text{(III.2.10)}$$

It is clear from Eq.III-2.10 that $K(w,t) \rightarrow H(w)$ as $t \rightarrow \infty$. The time dependent mean square response is given by putting $t_1 = t_2$ in the Eq. III-2.6.

$$E[y^2(t)] = \sigma^2_{y(t)} = \int_{-\infty}^{\infty} S_{xx}(w)\, K(w,t)\, K^*(w,t)\, dw \qquad \text{(III.2.11)}$$

Recalling the stationary case for which the mean square value is given by

$$E[y^2(t)] = \int_{-\infty}^{\infty} S_y(w)\, dw \qquad \text{(III.2.12)}$$

and comparing Eq.III-2.12 with III.2.11 we can write

$$S_y(w,t) = S_{xx}(w)\, K(w,t)\, K^*(w,t)$$
$$= S_{xx}(w) |K(w,t)|^2 \qquad \text{(III.2.12a)}$$

which is the time dependent power spectral density.

Equation III.2.11 after performing the integration gives the mean square response as

$$E[y^2(t)] = \sigma^2_y(t)$$
$$= \frac{\pi S_{xx}(0)}{2\zeta w_o^3}\left[1 - \frac{e^{-2\zeta w_o t}}{1-\zeta^2}\left(1-\zeta^2 \cos 2 w_d t + \zeta\sqrt{1-\zeta^2} \sin 2 w_d t\right)\right] \qquad \text{(III.2.13)}$$

Eq.III.2.13 can be further simplified by neglecting the higher order terms of ζ, as $\zeta << 1$, we get the final expression,

$$\sigma^2_{y(t)} = \left[1 - e^{-2\zeta w_o t}\right] \frac{\pi\, S_{xx}(0)}{2\zeta w_o^3} \qquad \text{(III.2.14)}$$

where

$S_x(0)$ is the uniform power spectral density of white noise.

III.3 Numerical Approach

The transient response of the single degree of freedom system, can also be evaluated, by finding directly the ensemble averages of the sample response functions. It may be written as

$$\sigma_y^2(t) = E[y(t)\ y(t)] \qquad \text{(III.3.1)}$$

$$= \int_{-\infty}^{\infty} y^2 p_Y(y)\ dy$$

where $P_Y(y)$ is the probability density function of the random variable, Y and is read as, probability of Y lying in the range y, y+dy.

Discretization of the Eq. III.3.1 gives,

$$\sigma_y^2(t) = \sum_{i=1}^{N} y_i^2 .\ P_y(y_i) \qquad \text{(III.3.2)}$$

As the probability density function is not known before hand, $\sigma_y^2(t)$ can be calculated by taking the arithmetic mean of y_i^2 as follows

$$\sigma_y^2(t) = \sum_{i=1}^{N} \frac{y_i^2(t)}{N} \qquad \text{(III.3.3)}$$

where i is taken across the ensemble.

III.4. Nonlinear Response Analysis

A single degree of freedom system of Duffing's type is considered for the nonlinear response analysis. The characteristic equation for such a system may be written as:

$$\ddot{y}(t) + 2\zeta w_o \dot{y}(t) + w_o^2 y(t) + \epsilon w_o^2 y^3(t) = F(t) \qquad \text{(III.4.1)}$$

For small ϵ and for the white noise excitation theoretical results are available and it can be shown that the mean square value of $y(t)$ is given by

$$\sigma_y^2 = \frac{\sigma_o^2}{(1+3\epsilon\sigma_y^2)} \qquad \text{(III.4.2)}$$

where σ_o^2 is the mean square value of the response $y(t)$ of corresponding linear system.

In the present thesis work the Monte-Carlo technique is employed to investigate the effect of varying ϵ on the system response characteristics.

III.5. Monte-Carlo Response Analysis of an Aircraft Landing Gear Assembly to Taxiing Induced Vibrations

Finally, to demonstrate the usefulness of the Monte-Carlo approach, an idealised model of the aircraft and its landing gear assembly is considered and its response is studied, to the taxiing induced vibrations due to runway roughness. Fig. III.2 shows a sketch of the model. In this model, the

landing gear mass is neglected as it is very small when compared to the mass of the aircraft. Following are the equations of motion of such a system. For the derivation Ref. [26] may be referred.

$$m_1 \ddot{x}_1 + c_1(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2) = 0 \qquad \text{(III.5.1(a))}$$

$$-k_1(x_1 - x_2) + k_2 x_2 - c_1(\dot{x}_1 - \dot{x}_2) = k_2 y(t) \qquad \text{(III.5.1(b))}$$

where M_1 is the mass of the aircraft, c_1 the damping coefficient, k_1 is the stiffness of the landing gear, k_2 the tyre stiffness. k_1, k_2 and c_1 are the parameters of an equivalent linear system of the aircraft landing gear assembly. For the details of the equivalent linearization technique employed, Ref. [18] may be referred. $y(t)$ is the runway roughness the power spectral density of which is given by Eq.II.4.2.

Eqs. III.5.1 can be split up into three first order differential equations of the following form:

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \frac{k_2}{m_1}[y(t) - y_3] \\ \dot{y}_3 &= 1/c_1 \,[k_2 y(t) - k_1(y_1 - y_3) - k_2 y_3 + c_1 y_2] \end{aligned} \qquad \text{(III.5.2)}$$

where $y_1 = x_1$; $y_2 = \dot{x}_1$; and $y_3 = x_2$ (III.5.3)

Initial Conditions:- At time $t = 0$, the system is at rest.

$$y_1 = 0$$
$$y_2 = 0 \quad \text{at} \quad t = 0 \qquad \text{(III.5.4)}$$
$$y_3 = 0$$

Once the time history of y(t) is simulated, the Eqn.III.5.3 can be integrated with the initial conditions given in Eq.III.5.4 and various response parameters such as, the r.m.s values of displacement of the aircraft mass, the deflection of tyre, the velocity of the aircraft mass and the acceleration of the aircraft mass can be computed.

III.5 Results and Discussion

The following structural systems are considered:- 1) A single degree of freedom system excited by band limited white noise. 2) A nonlinear system of Duffing's type excited by band limited white noise 3) An idealized model of an aircraft and its landing gear subjected to taxiing and take-off induced vibrations due to runway roughness. The results are divided into two broad sections. In the first one the results of Monte-Carlo approach are discussed and in the second one the results of transient response calculations based on ensemble averages are presented.

1) Monte Carlo Response Analysis

Two simple oscillators with critical damping ratios $\zeta = 0.1$ and $\zeta = 0.01$ are considered for the linear case. Sample

time history of the band limited white input process is simulated digitally on the computer and response is computed by integrating the equation of motion using a fourth order Runge-Kutta integration scheme. The step length of the integration is chosen as one fourtieth of the undamped natural period of the oscillator. Integration is carried out for 250 cycles of the response. Mean and Mean square values of the response are computed by taking the temporal averages over every 2.5 cycles. Results are presented in the form of a plot between the Mean square value and time in Fig.III.3, III.4. The dotted line in the figures correspond to the stationary value of the response obtained from the theoretical result. There is an excellent agreement between the simulated and theoritical stationary values for case $\zeta = 0.1$. The maximum error is about 5% . For the Case $\zeta = 0.01$, there is a large difference between the simulation and theoritical results and the response takes significantly long time to attain a stationary value.

Results of the nonlinear response analysis are presented in Table III.1. Fig.III.5 shows the mean square response of the Duffing's systems for four values of ϵ. . Also presented in the same figure is the result of a linear system for comparison. The dotted line in each case represents the stationary value of mean square response obtained by Eq.III.4.2. It may be inferred from the result that as the amount of non-

linearity increases the mean square response reduces, and there is a significant reduction in the time taken by the response to reach stationarity. For $\epsilon = 0.2$, the response has reached a stationary value after 15 cycles of response, whereas for $\epsilon = 0.1$, the system has not achieved stationarity even after 50 cycles of response. It may be concluded that the greater the amount of nonlinearity the lesser will be the number of cycles of response needed for computing the temporal averages.

The mean square response in the transient stage presented in the same figure does not mean anything as the transient response is non-stationary and an ensemble average should be considered in place of temporal average.

2. Transient Response Analysis

During the transient stage the response is not ergodic and an ensemble average should be considered in place of temporal average. An ensemble of 36 sample response functions for the case $\zeta = 0.1$ and 56 samples for the case $\zeta = 0.01$ are considered. Results are presented in Figs.III.6 and III.7. Also presented in the same figures are the results of Chakrovorty[10] for comparison purposes. The simulation results are smoothed by taking moving average over a length of 1 sec. or 5 cycles of response and plotted to the same scale. It is clear from the figure III.7 that there is a good

agreement between the theoritical and simulation results for the case $\zeta = 0.1$. However, for the case $\zeta = 0.01$, the simulation results are not so close to the theoritical results. The maximum error is about 25%. For the case $\zeta = 0.1$ the response attains a stationary value very quickly, approximately after 4 cycles. For the case $\zeta = 0.01$, theoritically, the mean square response reaches a stationary value after 35 cycles, whereas the simulation results do not show any stationarity in the range considered. To increase the accuracy more number of sample response functions should be considered and it is advisable to consider more number of cosine terms for the generation of sample input function.

CHAPTER IV

FIRST PASSAGE PROBABILITY AND OTHER STATISTICS

IV.1 For structural systems operating in random environment, it becomes necessary to know certain statistics, such as the first excursion time of a safe limit, 2) the amount of time spent above the safe domain and 3) the number of threshold crossings during the period of operation. Such statistics help in establishing the reliability and fatigue life of the structure. Many authors [13,14,15,16,17] have studied these problems and obtained solutions both analytically and numerically under certain basic assumptions. This chapter provides a discussion of the numerical approach to the problem using the results of simulation outlined in Chapter II. The results are compared with those of the former authors.

IV.2. First Passage Probability Density

First passage probability density, $p_x(T)$ may be defined as the probability that the random process $x(t)$ exceeds a threshold for the first time, during the time interval T to $T+dt$.

A simple linear oscillator, excited by a white random process is considered in the following discussion. For such a system it has been proved [2] assuming poison process for the arrivals of peaks, that

$$p_x(T) = A\alpha e^{-\alpha T} \qquad \text{(IV.2.1)}$$

where α is expected number of crossings of barrier b and is given by

$$\alpha = \nu_o \, e^{-b^2/2\sigma^2} \qquad \text{(IV.2.2)}$$

σ is steady state root mean square value of the response. A is equal to 1 for $b >> \sigma$.

ν_o is number of '0' crossings.

Chandiramani and Cook [13] have found out $p_x(T)$ numerically from the statistics of safe intervals, shown in Fig. IV.1. The excitation is simulated following the approach of Cook [23]. The probability density function $p_x(T)$ may be written as

$$p_x(T) = P_{i.f.}\, \delta(T) + (1 - P_{i.f.}) \int_T^\infty \frac{p(L)\, dL}{E[L]} \qquad \text{(IV.2.3)}$$

where $P_{i.f.}$, stands for probability of instantaneous failure under the stationary starting conditions. For '0' start condition clearly $P_{i.f.} = 0$ and Eq. IV.2.3 may be written as

$$p_x(T) = \int_T^\infty \frac{p(L)dL}{E[L]} \qquad \text{(IV.2.4)}$$

In the present thesis work a 'D' type barrier. [Double barrier $\pm$ b] with '0' start condition is assumed. A single degree of freedom linear oscillator subjected to band limited white noise is considered. A computer programme is developed which computes the response of the oscillator till it reaches, for the

first time, a threshold. The first passage time is stored in a dimensioned variable and the whole procedure is restarted. A separate subroutine simulates the random force. 1795 such first passage times are collected using five different random number generators.

The probability density function is not unique as it depends upon the choice of end points of the interval for the range of the variable and the choice of the number K of class intervals in which the range is divided. The optimum values of K for different sample sizes N are given in Table 4.1 of Section 4.6. of ref. [22]. K is chosen accordingly. The probability density function is estimated at the mid points of the K class intervals using the method outlined in Section 7.2.2, of the same reference.

IV.3 Expected Amount of Time Spent Above a Threshold

Consider a narrow band random process x(t), Fig. IV.1, such as the response of a lightly damped linear oscillator to white noise. The shaded area in the Fig. IV.1 corresponds to the time spent above a level 'b'. It is required to find out the ratio $\frac{\Sigma \Delta t_i}{T}$ which directly gives the expected amount of time spent per unit time above level 'b'. This can be evaluated by monitoring the output of the single degree of freedom system to a broad band excitation with the help of a digital computer.

Analytically it can be calculated as follows. Define a random process z(t) as

$$z(t) = 0 \quad \text{for} \quad x(t) < b$$
$$= 1 \quad \text{for} \quad x(t) \geq b \qquad \text{(IV.3.1)}$$

Now $\Sigma \Delta t_i$ is given by

$$\Sigma \Delta t_i = y = \int_0^t z(t)\, dt \qquad \text{(IV.3.2)}$$

clearly y is a random variable for which the expected value can be calculated by

$$E[y] = E \int_0^t z(t)\, dt = \int_0^t E[z(t)]\, dt \qquad \text{(IV.3.3)}$$

$$= \int_0^t \{(0) \times P[X(t) < b] + 1 \times P[X(t) \geq b]\}\, dt$$

$$= \int_0^t \int_0^\infty p_X(x,t)\, dx\, dt \qquad \text{(IV.3.4)}$$

As the system response is a Gaussian random process its probability density function is given by

$$p_X(x,t) = \frac{1}{\sqrt{2\pi}\,\sigma_x(t)} \exp\left(-x^2/2\sigma_x^2(t)\right)$$

where $\sigma_x^2(t)$, transient mean square response and

$$E[y] = \int_0^t \int_0^\infty \frac{1}{\sqrt{2\pi}\,\sigma_x(t)} \exp\left[-x^2/2\sigma_x^2(t)\right] dx\, dt$$

$$= \frac{1}{2} \int_0^t \operatorname{erfc}(\bar{z})\, dt \qquad \text{(IV.3.6)}$$

where $\bar{z} = \sqrt{b^2/2\sigma_x^2(t)}$ (IV.3.7)

Eq. IV.3.6 can be integrated using the standard procedures available for the numerical integration.

IV.4 Expected rate of crossings of a threshold:

The expected rate of upcrossings of a level 'b' in a narrow band Gausian random process, such as the response of a simple linear oscillator excited by a broad band random process is given by [2]

$$\nu^+(b,t) = \int_0^\infty \eta\, f_{x\dot{x}}(b,\eta;t)\, d\eta \qquad \text{(IV.4.1)}$$

where $f_{x\dot{x}}(b,\eta;t)$ is the joint probability density function of $x(t)$ and $\dot{x}(t)$ of the response. For the transient state it is given by

$$f_{x\dot{x}}(\xi,\eta,t) = \frac{1}{2\pi\sigma_x\sigma_{\dot{x}}\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{\xi^2}{\sigma_x^2} - \frac{2\rho\xi\eta}{\sigma_x\sigma_{\dot{x}}} + \frac{\eta^2}{\sigma_{\dot{x}}^2}\right)\right]$$

(IV.4.2)

where $\sigma_x^2 = E[x^2(t)]$

$\sigma_{\dot{x}}^2 = E[\dot{x}^2(t)]$

and $\rho = E[x(t)\,\dot{x}(t)]/(\sigma_x\sigma_{\dot{x}})$

Substituting Eq. IV.4.2 in Eq. IV.4.1 and carrying out the integration, it can be shown that

$$\nu^{+}(b,t) = \frac{1}{2\pi}\frac{\dot{\sigma}_x}{\sigma_x}\sqrt{1-\rho^2}\left\{\exp\left(\frac{-b^2}{2\sigma_x^2(1-\rho^2)}\right) + \sqrt{\pi}\, g(1+\operatorname{erf} g)\exp\left(\frac{-b^2}{2\sigma_x^2}\right)\right\} \qquad \text{(IV.4.2)}$$

where $$g = \frac{b}{\sigma_x\sqrt{2(1-\rho^2)}} \qquad \text{(IV.4.3)}$$

$$\sigma_x^2(t) = \frac{\pi S_o}{2\zeta w_o^3}\left[1 - \frac{\exp(-2\zeta w_o t)}{1-\zeta^2}\left\{1 - \zeta^2\cos 2w_d t + \zeta\sqrt{1-\zeta^2}\sin 2w_d t\right\}\right] \qquad \text{(IV.4.4)}$$

$$\dot{\sigma}_x^2(t) = \frac{\pi S_o}{2\zeta w_o}\left[1 - \frac{\exp(-2\zeta w_o t)}{1-\zeta^2}\left\{1 - \zeta^2\cos 2w_d t - \zeta\sqrt{1-\zeta^2}\sin 2w_d t\right\}\right] \qquad \text{(IV.4.5)}$$

$$\rho(t) = \frac{\pi S_o}{2\zeta w_o^3(\sigma_x\dot{\sigma}_x)}\left[\frac{w_o}{1-\zeta^2}\exp(-2\zeta w_o t)\,(1-\cos 2w_d t)\right]$$

S_o is the power spectral density of the input process and w_o the natural frequency of the oscillator.

Numerical approach:

Numerically it can be calculated as follows:

A sample record of considerable length is computed and is divided into a number of intervals. In each interval the actual number of crossings of a specified level 'a' is counted and its ratio to the interval length gives directly the number of crossings per unit time. This will be a random variable as we have considered only one record. To get the expected value a number of such samples should be considered and an average across the ensemble should be taken in a similar way in which the transient response is computed. In the present thesis work only six samples are considered.

IV.5. Results and Discussion

The response of a simple linear oscillator with critical damping ratio $\zeta = 0.1$ is considered for the analysis of first passage and the related statistics. For the calculation of first passage probability 1795 sample response functions are considered. The safe domain is of double barrier type. A computer programme is developed, which automatically collects the first passage times, by noting the time at which for the first time the absolute value of the response exceeds 1σ level. Once it crosses the procedure is restarted and a new sample is generated. The total computer time taken for collecting 1795 samples is about two and half hours. The temporal range is taken as 4 natural periods and

is divided into a large number of class intervals. First passage probability density is calculated by deviding the number of samples falling in a particular interval by the interval length and is plotted against the nondimensional time in Fig. IV.2. Results obtained by Crandall, Chandiramani and Cook [13] are also plotted to the same scale in the same figure. Toland Yang also have obtained similar results [16].

The results show a good agreement with the results of the previous authors except in the range.42N to .64N where the probability densities obtained in this work differ significantly from the previous results.

In Figs. IV.3. and IV.4 expected amount of time spent above 1σ and 2σ levels respectively by the oscillator response is plotted against nondimensional time. The dotted line in the figures correspond to the theoretical result obtained by the Eq. IV.36. It is clear from the curves that for 1σ level, there is an excellent agreement between the theoretical and simulation results approximately after 40 periods of response onwards. During the transient stage the simulation results predict slightly lesser values than theoretical values. It may be said that the expected amount of time spent per unit time takes more time to attain a stationary value for the case of simulation results.

For 2 σ level, the discrepancy between the theoretical and simulation results is more pronounced. Simulation results slightly overestimate the stationary value the maximum discrepancy being about 6%. . However, in the transient region there is a large discrepancy in the simulated and theoretical values, the simulated values being quite low compared to the theoretical results.

In the Fig. IV.5, the expected rate of upcrossings of 1 σ and 2 σ levels in a narrow band random process. The results show a similar behaviour.

The discrepancy in the simulated results is mainly due to the inadequate sample size. Only six samples are considered for the calculations. Theoretically the averages should be taken across the ensemble for infinite sample functions, wherever the principle of ergodicity is not valid. This explains clearly the discrepancy between the theoretical and simulated results in the transient region, where the response is not ergodic.

CHAPTER V

RANDOM WALK MODEL

V.1 In the preceding chapters we discussed an approach to the stochastic process analysis in which the stochastic process of interest is simulated following its spectral representation and response characteristics are computed using the simulated time history. Throughout the analysis only the first and second moment functions are computed. In the present chapter a different approach, the Markov process model of the stochastic process, is discussed. A brief description of Markov process approach is given below.

V.2 The Markov process is characterized by a distinct property, that given precise knowledge of the present, the future is independent of past. Though this condition makes it highly restrictive, it happens that many physical processes exhibit the Markov property. The response of a linear single degree of freedom oscillator is one such example. Doob [25] first stated that the response, velocity, and displacement of the linear system excited by white noise form components of Markov vector. Caughey [19] has discussed its applicability to non-linear systems. Chandiramani and Cook [13] solved several first passage problems using the solution of Wang and Uhlenbech [15] to the Fokker-Planck equation which characterizes the

Markove Process. Recently Toland [16] proposed a random walk model which is applicable to a single degree of freedom system. For a non-white excitation the solution is yet unsolved. Herein we describe briefly the Random Walk Model of Toland [20] and extend it to a two degree of freedom system subjected to white noise-excitation.

IV.3 Random Walk Model

A general single degree of freedom system can be represented as

$$\ddot{y} + H(\dot{y},y) = F(t) \; ; \quad \dot{y}(0) = 0, \quad y(0) = 0 \qquad (V.3.1)$$

where F(t) is a stationary Gaussian white noise with mean and correlation

$$E[F(t)] = 0 \qquad (V.3.2)$$

$$E[F(t_1)\, F(t_2)] = 2\pi\, S_o\, \delta(t_2-t_1)$$

δ is Dirac delta function.

It is required to find the transient conditional probability of response $\dot{y}$, y, at any time t given that the system started from rest.

This can be obtained by discretizing the Chapman-Kolomogorov-Smoluchowski equation, for the two dimensional Markov process and may be written as [Ref. no 16]

$$P[\dot{y}_i, y_j, t_k/0,0,0] = \sum_m \sum_m P[\dot{y}_m, y_n, t_{k-1}/0,0,0]$$

$$x \, \Omega_2[\dot{y}_i, y_j, t_k/\dot{y}_m, y_n, t_{k-1}]$$

(V.3.3)

$P[\dot{y}_i, y_i, t_k/0,0,0]$ is the probability that the system is at $\dot{y}_i$, y_j in the phase plane, at time t_k given that the system is at rest to start with. Ω_2 is the second order transitional probability.

Eq.3.3 implies that all the possible states $\dot{y}_m$, y_n in the phase plane, at t_{k-1} have the possibility of reaching $\dot{y}_i$, y_j at time t_k. To reduce the complexity of the problem, it is assumed that once the velocity takes a definite value $\dot{y}_i$ then the displacement follows the deterministic relation $y_j = \dot{y}_i \Delta t$... The transitional probability function now reduces to

$$\Omega_1(\dot{y}_i, t_k/y_m, y_n, t_{k-1}) \; \delta \, (y_j - y_n), \; \dot{y}_m \Delta t$$

The second assumption is that the velocities can change only by small amounts. Consequently it is assumed that the velocity can either increase by one unit or decrease by one unit, with probabilities p, and q. Now Eq. V.3.3 reduces to

$$P[\dot{y}_i, y_j, t_k/0,0,0] = P[\dot{y}_{i-1}, y_j - \dot{y}_{i-1}\Delta t, t_{k-1}/0,0,0] \quad x$$

$$P_1[\dot{y}_{i-1},\ y_j - \dot{y}_{i-1}\Delta t,\ t_{k-1}] + P[\dot{y}_{i+1},\ y_j - y_{i+1}\Delta t,\ t_{k-1}/0,0,0]x$$

$$q[\dot{y}_{i+1},\ y_j - y_{i+1}\Delta t,\ t_{k-1}/0,0,0] \qquad (V.3.4)$$

Eq.V.3.4 means that the structure which is at A presently as shown in Fig.V.1 was either at B and went up with probability p or it was at C and dropped down to A with probability q.

The relations for p and q are given by

$$p = \frac{1}{2}\left[1 - H(\dot{y}_i,\ y_j)\,\Delta t/\Delta^*\right], \qquad (V.3.5)$$

$$q = 1-p$$

and $$\Delta^* = \sqrt{2\pi\, S_o \Delta t}$$

Eqs. V.3.4, and V.3.5 characterizes completely the random walk model of the simple degree of freedom system. However, for the computational convenience Eq. V.3.4 may be written as

$$P(m,r,s) = P(m-1,\ r-m+1, s-1)\ p(m-1,\ r-m+1,\ s-1) +$$
$$+ P[m+1,\ r-m-1,\ s-1]\ q(m+1,\ r-m-1,\ s-1) \qquad (V.3.6)$$

where r, m, s are integer values representing discrete points in the phase plane as $m\Delta^* = \dot{y}_i$, $r\Delta\bar{y} = y_j$, and $s\Delta t = t_k$ and $\overline{\Delta y}$ is defined as $\Delta^* \cdot \Delta t$.

V.4. First Passage Probability Density

Random walk model can be applied to calculate the First passage probability density. Let us define $\bar{p}(\dot{y},y,t/0,0,0,\Gamma)$ or the probability that the response is $(\dot{y},y)$ at time t given that it is (0,0) at the start and all the probability masses are inside within the safe domain Γ. It can be shown that the probability density function of first passage time is given by [16]

$$g(t,\Gamma) = -\frac{\partial}{\partial t} \int_{\Gamma} \bar{p}(\dot{y},y,t/0,0,0,\Gamma)\, d\dot{y},\, dy \qquad (V.4.1)$$

Numerically it can be calculated by the finite difference approximation.

$$g(t,\Gamma) = \left[\frac{P(t) - P(t+\Delta t)}{\Delta t}\right] \qquad (V.4.2)$$

where

$P(t + \Delta t)$ is the total probability mass at time $t + \Delta t$ and $P(t)$ is the total probability mass at time t. within the domain .

V.5. Random Walk Model for a Two Degree of Freedom System

A general two-degree of freedom system may be represented as shown in Fig.V.3. It can be represented by the following equations of motion. $m_1\ddot{x}_1 + k_1(x_1-x_2) + c_1(\dot{x}_1 - \dot{x}_2) = F_1(t)$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2-\dot{x}_1) + (k_1+k_2)x_2 - k_1x_2 = F_2(t) \qquad (V.5.1)$$

where $F_1(t)$ and $F_2(t)$ are independent white noise excitations with '0' mean. Eqs. V.6.1 are second order coupled simultaneous differential equations. It is assumed here that the equations can be uncoupled using the normal mode approach [1]. The uncoupled equations may be written as,

$$\ddot{y}_1 + H(\dot{y}_1, y_1) = f_1(t)$$
$$\ddot{y}_2 + H(\dot{y}_2, y_2) = f_2(t) \qquad \text{(V.5.2)}$$

where $H(\dot{y}_1, y_1) = 2\zeta_1 w_1 \dot{y}_1 + w_1^2 y_1$

$H(\dot{y}_2, y_2) = 2\zeta_2 w_2 \dot{y}_2 + w_2^2 y_2$

$f_1(t)$ and $f_2(t)$ are white noise excitations having the following characteristics.

$$E[f_1(t)] = 0, \quad E[f_2(t)] = 0$$
$$E[f_1(t), f_1(t+\tau)] = 2\pi S_1 \delta(\tau) \qquad \text{(V.5.3)}$$
$$E[f_2(t)\, f_2(t+\tau)] = 2\pi S_2 \delta(\tau)$$

It is required to find the transient probability of response $\dot{y}_1$, y_1, $\dot{y}_2$ at any time t, given that the system started from rest. Discretisation of the Chapman, Komogorov - Smoluchowski equation for the above system gives directly the transient probability of response as

$$P[\dot{y}_{1i}, y_{1j}, \dot{y}_{2k}, y_{2l}, t_m/0,0,0,\bar{0}] =$$

$$\sum_n \sum_o \sum_p \sum_q P[\dot{y}_{1n}, y_{1o}, \dot{y}_{2p}, y_{2q}, t_{m-1}/0,0,0,\bar{0}]$$

$$x \Lambda_4(\dot{y}_{1i}, y_{1j}, \dot{y}_{2k}, y_{2l}, t_m/\dot{y}_{1n}, y_{1o}, \dot{y}_{2k}\, y_{2q}, t_{m-1}) \qquad (V.5.4)$$

Under the same assumptions as that of the single degree of freedom system, the fourth order transient probability function Λ_4 can be reduced to a second order transient probability function Λ_2 and may be written as

$$\Lambda_4(\dot{y}_{1i}, y_{1j}, \dot{y}_{2k}, y_{2l}, t_m/\dot{y}_{1n}, y_{1o}, \dot{y}_{2p}, y_{2q}, t_{m-1})$$

$$= \Lambda_2(\dot{y}_{1i}, \dot{y}_{2k}, t_m/\dot{y}_{1n}, \dot{y}_{2p}, t_{m-1})\, \delta[y_{1j} - y_{1o}, \dot{y}_{1n}\Delta t]$$

$$x\, \delta[y_{2l} - y_{2q}, \dot{y}_{2p}\Delta t] \qquad \ldots(V.5.5)$$

Substituting Eq. V.5.5 in equation V.5.4

$$P[\dot{y}_{1i}, y_{1j}, \dot{y}_{2k}, y_{2l}, t_m/0,0,0,\bar{0}] = \sum_n \sum_p P[\dot{y}_{1n}, y_{1j} - \dot{y}_{1n}\Delta t,$$

$$\dot{y}_{2p}, y_{2l} - \dot{y}_{2p}\Delta t, t_{k-1}/0, 0, 0, 0, \bar{0}]\, x \Lambda_2[\dot{y}_{1i}, \dot{y}_{2k}, t_k$$

$$[\dot{y}_{1n}, y_{1j} - \dot{y}_{1n}\Delta t, \dot{y}_{2p}\, y_{2l} - y_{2p}\Delta t, t_{k-1}] \qquad (V.5.6)$$

Ω_2 is a second order transition probability density function and gives the probability that velocity changes from $\dot{y}_{1n}$, $\dot{y}_{2p}$ at t_{k-1} to $\dot{y}_{1i}$, $\dot{y}_{2k}$ at t_k.

Now we assume that the velocity changes are likely to be in the closest neighbourhood during a small interval of time t. Consequently it is assumed that in one time interval the velocity can either increase or decrease by one unit i.e.

$\dot{y}_{1n}$ can become y_{in+1}, or y_{1n-1} and

$\dot{y}_{2k}$ can become y_{2p+1}, or y_{2p-1}

So there are altogether 4 combinations associated with these changes.

Let

p_1 be the probability of increase of $\dot{y}_1$

q_1 be the probability of decrease of $\dot{y}_1$

p_2 be the probability of increase of $\dot{y}_2$

q_2 be the probability of increase of $\dot{y}_2$

Ω_4, the transient probability now reduces to

$$\Omega_2 [\dot{y}_{1i}, \dot{y}_{2k}, t_m/\dot{y}_{1n}, y_{1j} - \dot{y}_{1n}\Delta t, \dot{y}_{2p}, y_{2l} - \dot{y}_{2p}\Delta t, t_{m-1}$$

$$= \delta(\dot{y}_{1i}, \dot{y}_{1n+1})\, \delta(\dot{y}_{ik}, \dot{y}_{2p+1})\, p_1 p_2$$

$$+ \delta(\dot{y}_{1i}, \dot{y}_{1n-1})\, \delta(\dot{y}_{2k}, \dot{y}_{2p+1})\, q_1 p_2$$

$$+ \delta(\dot{y}_{1i}, \dot{y}_{1n+1})\, \delta(\dot{y}_{2k}, \dot{y}_{2p-1})\, p_1 q_2$$

$$+ \delta(\dot{y}_{1i}, \dot{y}_{1n-1})\, \delta(\dot{y}_{2k}, \dot{y}_{2p-1})\, q_1 q_2 \qquad \text{(V.5.7)}$$

Substituting V.5.7 in V.5.6.

$$P[\dot{y}_{1i}, y_{ij}, \dot{y}_{2k}, y_{2l}, t_m/0,0,0,0,0] =$$

$$P[\dot{y}_{1i+1}, y_{1j}, \dot{y}_{2k+1}, y_{2l}, t_m/0,0,0,0,0]\, q_1 q_2$$

$$+ P[\dot{y}_{1i-1}, y_{1j}, y_{2k+1}, y_{2l}, t_m/0,0,0,0,0]\, p_1 q_2$$

$$+ P[\dot{y}_{1i-1}, y_{1j}, y_{2k-1}, y_{2l}, t_m/0,0,0,0,0]\, p_1 p_2$$

$$+ P[\dot{y}_{1i+1}, y_{1j}, y_{2k-1}, y_{2l}, t_m/0,0,0,0,0]\, q_1 p_2 \qquad \text{(V.5.8)}$$

To calculate the one-step probabilities $p_1 p_2, q_1$ and q_2 we use the statistics of the velocity increments

$$\begin{aligned}
\langle \Delta \dot{y}_1 \rangle &= p_1 \Delta_1^* + q_1(-\Delta_1^*) \\
\langle (\Delta \dot{y}_1)^2 \rangle &= p_1 \Delta_1^{*2} + q_1(-\Delta_1^*)^2 \\
\langle \Delta \dot{y}_2 \rangle &= p_2 \Delta_2^* + q_2(-\Delta_2^*) \\
\langle (\Delta \dot{y}_2)^2 \rangle &= p_2 \Delta_2^{*2} + q_2(-\Delta_2^*)^2
\end{aligned} \qquad \text{(V.5.9)}$$

Relating the statistics to the system problems and excitation

by [1]

$$
\begin{aligned}
\langle \Delta \dot{y}_1 \rangle &= -H(\dot{y}_{1i}, y_{1j})\, \Delta t \\
\langle \Delta \dot{y}_2 \rangle &= -H(\dot{y}_{2i}, y_{2j})\, \Delta t \\
\langle \Delta \dot{y}_1^{\,2} \rangle &= 2\pi S_1 \Delta t \\
\langle \Delta \dot{y}_2^{\,2} \rangle &= 2\pi S_2 \Delta t
\end{aligned}
\qquad \text{(V.5.10)}
$$

Eqns. V.5.9 and V.5.10. provide four equations for the four unknowns p_1, p_2, q_1 and q_2 which are found to be

$$
\begin{aligned}
p_1 &= 1/2 \left[1 - H(\dot{y}_1, y_1) \frac{\Delta t}{\Delta^*_1}\right] \\
q_1 &= 1 - p_1 \\
p_2 &= \frac{1}{2} \left[1 - H(\dot{y}_2, y_2) \frac{\Delta t}{\Delta^*_2}\right]
\end{aligned}
\qquad \text{(V.5.11)}
$$

In the above formulation it is assumed that y_1 and y_2 are independent and their joint probability is equal to the product of the individual probabilities.

As a check to the velocity of the above model, the recurrence equation Eq. V.5.8 in the limit as $\Delta t \to 0$ can be shown to approach the Fokker-Planck differential equation.

V.6 First Passage Probability Density For the Two Degree of Freedom System

The random walk model derived above can be used to find the first passage probability density. Suppose we define the safe domain as $[x_1] \leq a$. The coordinates x_1, x_2, $\dot{x}_1$, $\dot{x}_2$ of the original system can be expressed in terms of the generalized coordinates by the following relation:

$$\bar{x} = [D]\,\bar{y} \qquad (V.6.1)$$

where $\bar{x} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{Bmatrix}$, and $\bar{y} = \begin{Bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{Bmatrix}$, and

D is a matrix.

The conditional probability of the generalised coordinates $P[\dot{y}_1, y_1, \dot{y}_2, y_2]$ can be found out from the random walk model. (Here the conditional sign is dropped for convenience)

Now $P[x_1, x_1, \dot{x}_2, x_3]$ can be expressed in terms of $P[\dot{y}_1, y_1, \dot{y}_2, y_2]$.

We have

$$\begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{Bmatrix} = [D] \times \begin{Bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{Bmatrix} \qquad (V.6.2)$$

or

$$\begin{Bmatrix} y_1 \\ \dot{y}_1 \\ y_2 \\ \dot{y}_2 \end{Bmatrix} = [D]^{-1} \times \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{Bmatrix} \qquad (V.6.3)$$

Hence $P[\dot{x}_1, x_1, \dot{x}_2, x_2] = P[\,|D|^{-1}\,\bar{x}\,]$

$$= P[\,\bar{y} = |D|^{-1}\,\bar{x}\,] \qquad (V.6.4)$$

which is known.

Now for any time t,

$$P[x_1 \leq a] = \sum_{\dot{x}_2} \sum_{x_2} \sum_{\dot{x}_1} \sum_{x_1 \leq a} P[\dot{x}_1, x_1, \dot{x}_2, x_2] \qquad (V.6.5)$$

First passage probability density.

$$g(t, \Gamma) = + \frac{[P(T) - P(T+\Delta t)]}{\Delta t} \qquad (V.6.6)$$

where $P(T)$ - Total probability mass at time $t = T$,

and $P(T+\Delta t)$- Total probability mass at time $t = T+\Delta t$ enclosed inside the safe domain defined by $x_1 \leq a$.

V.7 Numerical Aspects

Some of the numerical aspects such as the choice of proper values for Δt, Δ^* etc. and the numerical difficulties that may be encountered due to the memorary limitation of the computer are discussed in this section.

The phase plane is bounded in the $\dot{y}$ direction because of the natural boundary conditions $0 \leq p \leq 1$, which means

$$[H(\dot{y}_1 y)] \leq \frac{2\pi S_o}{\Delta t} \qquad (V.7.1)$$

For a simple linear oscillator Eq. V.5.1 reduces to

$$2\zeta w_o \dot{y} + w_o^2 y \leq \frac{2\pi S_o}{\Delta t} \qquad (V.7.2\text{'a})$$

$$-2\zeta w_o \dot{y} - w_o^2 y \geq \frac{2\pi S_o}{\Delta t} \qquad (V.7.2\text{ b})$$

Eq. V.5.2 represent a pair of straight lines as shown in Fig. V.2. dividing the phase plane into a sub domain. It is clear from the equations that only Δt need be specified for the numerical study. A proper choice should be made such that it gives a sufficiently large subdomain for a reasonable computational accuracy with a desired stability. The stability here means a finite mean and a finite variance for all time [27]. A single degree of freedom system with critical damping ratio $\zeta = 0.08$, $S_o = 2$ units, and $w_o = 5$ cps is considered here for the study. Absorbing boundaries are set at $\pm$ 1 distance in the y direction for the first passage probability density calculations. Any probability mass diffusing across these barriers is considered lost and will not take part in further diffusion. Though, the $\dot{y}$ direction is bounded to reduce the computational effort two artificial boundaries of absorbing type are set at $\pm$ 1.5 $\sigma_{\dot{y}}$, as shown in Fig. V.2. In choosing this, it should be kept in mind that the

probability mass diffusing accross these barriers should be of several orders of magnitude lesser than that lost accross displacement barrier.

Table V.1 presents how the memory requirements would vary with the choice of Δt. The number of cells into which the phase plane is to be divided increases enormously with the reduction of Δt and soon surpasses the memory limit of any computer. For the present work Δt = 1/125th of the natural period of oscillator is tried, however the process is found to be still unstable. Results show a large deviation from the theoretical results.* A minimum Δt of 1/500th of the natural period of the oscillator has been suggested by the authors of the random walk model. Accordingly the number of cells into which the phase plane is to be divided would be 120,000 and the memory needed would be approximately three times the total number of cells i.e. 360K. As the capacity of the existing computer is only 28K the problem can not be handled directly.

Use of tape units to store the value is also tried. But the time involved in the winding and unwinding operations is found to be too prohibitive. To overcome this sort of difficulty, the authors [20] have suggested an approach in which a number of cells amounting to approximately $\sigma_y/10$ are grouped together and in the $\dot{y}$ direction the grid structure is unchanged. Probability mass diffuses into each

* REFER Fig V.4

cell in each interval of time. Thus an averaging process occurs for the y direction. This is not tried in the present thesis work.

The problem for the case of two degree of freedom system is formulated but even attempts to solve this problem could not be made because of the unwieldy computer memory storage requirements which was felt even for the case of single degree of freedom system where the computer memory required was of the order of 100K.

CHAPTER VI

CONCLUSIONS

Sample functions of a stationary, Gaussian one one dimensional, and one variate random process, with a prescribed power spectral density are simulated digitally on the computer. The samples are found to satisfy the tests for stationarity. Also the power spectral density estimates from the samples are found to agree with the target values. Parzen's lag window is found to give a smoother and better estimate than Hamming's lag window.

The transient and stationary response of simple linear oscillator oscillators with the damping ratios $\zeta = 0.1$ and $\zeta = 0.01$ are computed. A moving average over a length of 5 cycles of response is taken to smoothen the transient response. It is found that for the case of $\zeta = 0.1$, the moving average values show a good agreement with the theoretically obtained results. For the case $\zeta = 0.01$, a larger sample size (only 50 samples are considered in the present work) is recommended to reduce the discrepancy between the results obtained in the present work and the theoretical results. For the stationary response a sample record of 150 cycles of response is found to be sufficient.

A few nonlinear systems of Duffing's type are also considered for the response analysis. The stationary r.m.s.

value of the response is computed based on the temporal average over a length of 50 cycles. It is found that with the increase of non-linearity there is a significant reduction in the time taken to reach stationarity.

First passage probability density is found out numerically and is smoothened by taking a moving average over a length of 0.1 cycle. The results show a reasonably good agreement with those of previous authors. The expected amount of time spent above 1σ and 2σ level and the expected rate of upcrossings of 1σ and 2σ level by the response of a simple oscillator are computed both analytically and numerically. The stationary values show an excellent agreement.

A random-walk model for a two degree of freedom system is formulated. However, it is found that the memory of the existing computer system is not sufficient to handle the problem.

TABLE NO. II.1

RESULTS OF TESTS FOR STATIONARITY

Level of Significance $\alpha = 0.1$

S.No.	Sequence	Observed Values		Theoretical bounds	
		No. of runs	No. of reverse arrange-ments	No. of runs	No. of reverse arrange-ments
1.	Means	22	417	15-26	319-460
2.	Mean square values	25	386		

Note: The theoretical bounds for the number of runs and the number of reverse arrangements are taken from the tables 4.11 and 4.12, pages 170-171, of reference [22].

TABLE NO. II.2

POWER SPECTRAL DENSITY FUNCTION OF THE SIMULATED SAMPLE FUNCTION

Data	Target power spectral density = 4 Frequency limits 0 cps to 100 cps Maximum lag number m = 20.

S.No.	Frequency in cps	P.S.D. Hanning's Lag Window	Parzen's lag window
1	0	4.13	4.05
2	5	4.06	4.09
3	10	4.06	4.04
4	15	4.16	4.07
5	20	4.18	4.08
6	25	4.13	4.06
7	30	4.03	4.03
8	35	4.09	4.06
9	40	4.24	4.12
10	45	4.41	4.19
11	50	4.52	4.19
12	55	4.13	4.07
13	60	3.83	3.96
14	65	3.88	3.93
15	70	3.84	3.91
16	75	3.73	3.86
17	80	3.59	3.82
18	85	3.63	3.82
19	90	3.72	3.85
20	95	3.75	3.87
21	100	3.78	3.88

TABLE III.1

STATIONARY RESPONSE VALUES OF THE

NONLINEAR SINGLE-DEGREE OF FREEDOM SYSTEMS

System Characteristics

Natural frequency of the corresponding linear system is 5 cps. ζ = 0.1.

Excitation power spectral density = 4 x 10^4 units.

S.No.	e	Stationary mean square value of the response	
		Simulation	Theoretical
1	0.00	10.50	10.10
2	0.01	7.54	8.08
3	0.05	5.77	5.5
4	0.1	4.41	4.37
5	0.2	2.97	3.35

TABLE NO.V.1

System characteristics

$\zeta = 0.08$, $w_o = 5$ cps $\quad E_o = 2$ units

S.No.	Δt	$\frac{1}{w_o \Delta t}$	Δ^*	$\bar{y}/\Delta^*$	$\bar{y}/\Delta y$	Appr.no. of memory locations needed
1	.004	50	0.224	5	40	2400
2	.002	100	0.1591	8	112	10740
3	.001	200	0.1122	10	317	57100
4	.0004	500	0.071	16	1250	360000

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